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Double inequality for Hyperfactorial and its application

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Dedicated to our friend Mihály Bencze on his 70th birthday

Let $H(n) := \prod_{k=1}^{n} k^k$ (hyperfactorial) and $sf(n) := 1!2!3!...n!$ (superfactorial).

Theorem.

$$(n!)^{n/2} \cdot e^{\frac{(n-1)n}{4}} \le H(n) \le (n!)^{(n+1)/2} \cdot e^{\frac{(n-1)n}{4}}, n \in \mathbb{N}.$$

Proof. Let $L(n) := (n!)^n \cdot e^{\frac{(n-1)n}{2}}$ and $U(n) := (n!)^{n+1} \cdot e^{\frac{(n-1)n}{2}}$.
Then

$$(n!)^{n/2} \cdot e^{\frac{(n-1)n}{4}} \le H(n) \le (n!)^{(n+1)/2} \cdot e^{\frac{(n-1)n}{4}} \iff$$

$$\iff L(n) \le H^2(n) \le U(n) \tag{1}$$

$\forall n \in \mathbb{N}$.
First we will prove that

$$\frac{L(n+1)}{L(n)} < \frac{H^2(n+1)}{H^2(n)} < \frac{U(n+1)}{U(n)} \tag{2}$$

any $n \in \mathbb{N}$.
We have

$$(2) \iff \frac{((n+1)!)^{n+1} \cdot e^{\frac{n(n+1)}{2}}}{(n!)^n \cdot e^{\frac{(n-1)n}{2}}} < (n+1)^{2(n+1)} < \frac{((n+1)!)^{n+2} e^{\frac{n(n+1)}{2}}}{(n!)^{n+1} e^{\frac{(n-1)n}{2}}} \iff$$

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$$\Leftrightarrow \begin{cases} (n+1)^n < n!e^n \\ n! \cdot e^n < (n+1)^{n+1} \end{cases} \iff \frac{(n+1)^n}{e^n} < n! < \frac{(n+1)^{n+1}}{e^n},$$

where the latter double inequality holds for any $n \in \mathbb{N}$.

Indeed, since $e < \left(1+\dfrac{1}{n}\right)^{n+1} = \dfrac{(n+1)^{n+1}}{n^{n+1}}, \forall n \in \mathbb{N}$ then

$$e^n < \prod_{k=1}^{n} \frac{(k+1)^{k+1}}{k^{k+1}} = \frac{1}{n!} \prod_{k=1}^{n} \frac{(k+1)^{k+1}}{k^k} = \frac{(n+1)^{n+1}}{n!} \iff n! < \frac{(n+1)^{n+1}}{e^n}$$

Inequality

$$\frac{(n+1)^n}{e^n} < n! \iff (n+1)^n < e^n \cdot n!$$

we will prove using Math induction.

For $n = 1$ we have $(1+1)^1 < 1!e^1 \iff 2 < e$ and

$$\frac{(n+2)^{n+1}}{(n+1)^n} < \frac{(n+1)!e^{n+1}}{n!e^n} = (n+1)\,e \iff \frac{(n+2)^{n+1}}{(n+1)^{n+1}} < e \iff$$

$$\iff \left(1+\frac{1}{n+1}\right)^{n+1} < e,$$

$\forall n \in \mathbb{N}$.

For any $n \in \mathbb{N}$ assuming $(n+1)^n < e^n n!$ we obtain

$$(n+2)^{n+1} = \frac{(n+2)^{n+1}}{(n+1)^n} \cdot (n+1)^n < \frac{(n+1)!e^{n+1}}{n!e^n} \cdot e^n n! = (n+1)!e^{n+1}.$$

Thus, by MI $(n+1)^n \le n!e^n, \forall n \in \mathbb{N}$.

Corollary.

$$\lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt[n^2]{sf\,(n)}} = e^{3/4}$$

Proof. Since $sf\,(n) = \dfrac{(n!)^{n+1}}{H\,(n)}$ and

$$(n!)^{n/2} \cdot e^{\frac{(n-1)n}{4}} \le H\,(n) \le (n!)^{(n+1)/2} \cdot e^{\frac{(n-1)n}{4}}, n \in \mathbb{N}.$$

then

$$(n!)^{n/2}\cdot e^{\frac{(n-1)n}{4}} \le \frac{(n!)^{n+1}}{sf(n)} \le (n!)^{(n+1)/2}\cdot e^{\frac{(n-1)n}{4}} \iff \frac{(n!)^{n/2}\cdot e^{\frac{(n-1)n}{4}}}{(n!)^{n+1}} \le$$

$$\le \frac{1}{sf(n)} \le \frac{(n!)^{(n+1)/2}\cdot e^{\frac{(n-1)n}{4}}}{(n!)^{n+1}} \iff \frac{e^{\frac{(n-1)n}{4}}}{(n!)^{(n+2)/2}} \le$$

$$\le \frac{1}{sf(n)} \le \frac{e^{\frac{(n-1)n}{4}}}{(n!)^{(n+1)/2}} \Leftrightarrow \left(\frac{e^{\frac{(n-1)n}{4}}}{(n!)^{(n+2)/2}}\right)^{1/n^2} \le$$

$$\le \frac{1}{\sqrt[n^2]{sf(n)}} \le \left(\frac{e^{\frac{(n-1)n}{4}}}{(n!)^{(n+1)/2}}\right)^{1/n^2} \iff \frac{e^{\frac{n-1}{4n}}}{(n!)^{(n+2)/2n^2}} \le$$

$$\le \frac{1}{\sqrt[n^2]{sf(n)}} \le \frac{e^{\frac{n-1}{4n}}}{(n!)^{(n+1)/2n^2}} \iff \frac{e^{\frac{n-1}{4n}}\sqrt{n}}{(n!)^{(n+2)/2n^2}} \le$$

$$\le \frac{\sqrt{n}}{\sqrt[n^2]{sf(n)}} \le \frac{e^{\frac{n-1}{4n}}\sqrt{n}}{(n!)^{(n+1)/2n^2}}.$$

We have

$$\lim_{n\to\infty}\frac{e^{\frac{n-1}{4n}}\sqrt{n}}{(n!)^{(n+2)/2n^2}} = e^{1/4}\sqrt{\lim_{n\to\infty}\frac{n}{(n!)^{\frac{1}{n}+\frac{2}{n^2}}}} = e^{1/4}\cdot e^{1/2} = e^{3/4}$$

because

$$\lim_{n\to\infty}(n!)^{\frac{1}{n^2}} = \lim_{n\to\infty}\sqrt[n]{\frac{\sqrt[n]{n!}}{n}}\cdot\sqrt[n]{n} = 1.$$

And $\lim\limits_{n\to\infty}\dfrac{e^{\frac{n-1}{4n}}\sqrt{n}}{(n!)^{(n+1)/2n^2}} = e^{1/4}\sqrt{\lim\limits_{n\to\infty}\dfrac{n}{(n!)^{\frac{1}{n}+\frac{1}{n^2}}}} = e^{3/4}$ by the same reason.

Thus, by Squeeze theorem $\lim\limits_{n\to\infty} \dfrac{e^{\frac{n-1}{4n}}\sqrt{n}}{(n!)^{(n+2)/2n^2}} = e^{3/4}$.

REFERENCE

[1] Octogon Mathematical Magazine (1993-2024)

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